Plausible Reasoning and The Analogy in Posing and Solution of A Problem For A Mathematics Competition

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Abstract

When discussing how the mathematical thinking in the process of posing a problem is, in particular of the magic square generator, it is being investigated in the so called metacognition. The same problem is applied to a group of 32 students in a training of the Colombian Math Olympics in 2016 as a qualifying test and analyzed all solutions; as well as the associated cognitive processes. Identified that the both processes lead to contrasted sensibility. It is very interesting to show as a plausible reasoning presented, perpetually, in the process of creating the problem and analogy in all the solutions.

Key word: solving problem, posing problem, plausible reasoning, analogy, conjecture and insight.

Instruction

The study of the cognitive process about posing problems has been object of investigations in math education in the last years (Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B. 2005; English, Fox, & Walter, 2005; Priest, 2009; Kar, T., Özdemir, E., Sabri İpek, A., & M. Albayrak, 2010).

The study of analogical reasoning has also motivated the interest to research since that Polya and his followers begin to publish their works in 1933 (Polya, 1964, Lakatos, I., 1976).

Although the creation of mathematical problems has been much studied, since 1986 until the current, several papers have been published about. The paper by Engel (1987) "The creation of mathematical Olympiad Problem" is in force yet.

The teaching of mathematics based on problem solving is a topic repeatedly subject in all ICMEs. Within this general frame, much authors (Bellot, F.,2017) consider in these publications an approach to problems creation that in this paper could name "From a mathematical motivation to a problem".

In other hand, (Falk, M., 2017) in her important contribution says that Worrall (Lakatos, I., 1976) described this work in the following terms. The thesis of 'Proofs and Refutations' is that the development of mathematics does not consist (as conventional philosophy of mathematics tells us it does in the steady accumulation of eternal truths.

Polya and Lakatos attack the formalism of Hilbert, although it is worth while noting that Hilbert himself always recognized the importance of to attract young minds to mathematics, and his famous list of 23 problems in the International Congress of Mathematicians in Paris 1900 is a proof.

Mathematics develops, according to Lakatos, in a much more exciting way - by a process of conjecture, followed by attempts to prove the conjecture (i.e. to reduce it to other conjectures) followed by criticism via attempts to produce counter-examples both to the conjectured theorem and to the various steps in the proof.

It is important to remark what is analogy as a thinking process; this can be described as a thinking mechanism, as a way of thinking and even as some kind of similarity. The essence of analogical reasoning lies in the identification and transfer of structures and relationships from a well-known system (from the source) to a less well-known system (to the target).

The analogy requires maintenance, handling, activation and selective inhibition of mental representations, aimed to establish correspondences and inferences about similarity relations of higher order.

Other concept of this work is the plausible reasoning as a thinking process. There are two types of reasoning in math, the demonstrative and the plausible. It is important to distinguish a demonstration of an intuition, a valid an attempt without validity test, while plausible reasoning is important to distinguish between intuitions ones more and others less reasonable.

A plausible reasoning is present when through of proofs of essay and error until secure a conjecture or find a validity proof. The concept of conjecture in mathematics refers to a statement that assumed some, but that still has not been demonstrated or refuted. When the veracity of a conjecture is shown, then it is a true proposition.

The objective this researching is contrasting the thinking types in the posing and solving of a math problem. So, it is organized in two parts.

Results

Part I. Toward an approach to algorithms of solution to a square magic problem

In a presentation of UAN's Math Education postgraduate programs Science Fair in Bogota, some years ago a colleague presented the problem:

Found a distribution the firsts 12 natural number, such that, divided in two rows and six columns the sums of all columns and rows are equals respectively.

The solution was found very fast. Firstable, the distribution form in zigzag, as show:

2		0			
	1				

The sum of all columns was 13. The rows summed 42 and 36 respectively, the difference was 6, so that, it was necessary exchange a cell to correct the sums. If is swapped the number 5 by the 8, the two rows are equals.

2		0			
	1				

Now both rows sum 39, of course, the columns continue invariants. This moment was very motivator and this paper was continued with analogous problems.

When was begun with the sequence {6n}, to n=1 the sum of the first 6 natural numbers is 21; but this case is not possible, because 21 isn't divisible by 2 and can´t exist two rows with sums equals. Therefore, if the general term of the sequence is divisible by 4, it could ensure that this property is a necessary condition. If it can prove, then we are facing a valid conjecture.

It is valid generalize this problem and to begin to analyze the sequence {4n} with n=1 for the firsts 4 natural numbers. When these are organized in zigzag is easy to see the problem does not have solution, because there are not exchange possible of cells between two rows that could equal their sums.

With the sequence {4n} to prove with n≥2 and to exclude to n=1, of course, and to distribute in zigzag form, as show:

The sum of the 8 firsts natural number is 36 and the two rows must be equals; so that, each row must be the half, indeed 18. The sum each columns is 9. But, the first row sum 20 and the second 16. To correct the difference is necessary exchange some cells. If some cells are swapped between two rows, the sums of the columns will continue being the same. The problem now is answering the question. How many values of first row we have exchange to second row for that the sums could be equals?

After of much proofs it is found that the exchange of 2 firsts or 2 lasts cells, the sum both rows will be equals, as follow:

or

This work continues with n= 4, P (4)=16 and these number are distributed in two rows in zigzag form, its sum is 136 and the rows must be equals, each row must sum 68, however, these rows sum 72 and 64, respectively:

6		4		2		0	
	5		3		1		

It is necessary to exchange some cells to correct the difference; so that, if it is exchanged cells of the first row to the same position in the second row; then, the sums of all columns will be the same again. This exchange with 2 cells has failed; however, if exchange 4 cells, then the property is obtained with 3 solutions.

6			3		1	0	
	5	4		2			

Or

	5		3	2		0	
6		4			1		

Or

6		4			1		
	5		3	2		0	

too

With n=5, P (5) = 20. If It is arranged in zigzag form, as follow:

0		8		6		4		2	0
	9		7		5		3		1

As sum of the 20 firsts natural number is 210 and the two rows must be equals; so that, each row must be the half, indeed 105. The sum each columns is 21. But, the first row sum 110 and the second 100. To correct the difference is necessary exchange some cells. So that, if it is made swap the 3 cells, then is obtained:

0		8		6			3		0
	9		7		5	4		2	1

Now, both sums the two rows are equals. But, is it necessary would be 3 cells? It is yes, 14, 8, 20 of the row 1 by 7, 13, and 9 of the row 2

The calculi continue now for n=6 with 24 numbers, of same manner:

4		2		0		8		6	0	4	2
	3		1		9		7		5	1	3

The sum of the 24 firsts natural number is 300 and the two rows must be equals, so that each row must be the half, indeed 150. As the sum each columns is 25 and these are invariants. The first row sum 156 and the second row 144. To correct the difference is necessary exchange some cell. The proof with 4 cells failed; however, but if the exchange is made with 6 cells, then the property is obtained, as follow:

4		2		0			7		5	1	3
	3		1		9	8		6	0	4	2

Now, both sums of the two rows are equals. But, is it necessary the last 6 cells? It is not necessary. It is possible to obtain other 3 solutions.

To continue with n=7 and 28 natural numbers:

8		6		4		2		0	0	8	2	6	4
	7		5		3		1		9	1	7	3	5

The sum of the 28 firsts natural number is 406 and the two rows must be equals, so that each row must be the half, indeed 203. The first row sum 210 and the second row 196. To correct the difference is necessary exchange some cell.

So, that, we must exchange some cells between the rows. How many cells are necessary exchanges? To proof with 6 cells is failed; however, if it exchanges with one specific cell, the 18 with the 11, then is obtained the property, as follow:

8		6		4		2		0	0	1	2	6	4
	7		5		3		1		9	8	7	3	5

The proof to n=8 with 32 numbers, in analogous form is:

2		0		8		6		4	0	2	2	0	4	8	6
	1		9		7		5		3	1	1	3	9	5	7

The sum of the 32 firsts natural number is 528 and the two rows must be equals, so, each row must be the half, indeed 264. The first row sum 272 and the second row 256. To correct the difference is necessary exchange some cell between the rows. It was known that 8 cells it could obtain the property, as follow:

2		0			7		5		3	1	1	0	4	8	6

	1		9			6		4	0	2	2	3	9	5	7

When the proof is done the first 36 natural numbers and it is compared with the cases before, the property is not achieved. It do not exist cells to exchange and rectify 9 units, because only is possible to rectify difference when the units are an even number.

As summary the sequence {4n} to n=1, 3, 5, 7, 9 must be exclude and to n=2, 4, 6, 8 are:

- n=2 and 8 first natural numbers were necessary swap with the 2 first or last cells, 2 solutions possible.
- n=4 and 16 first natural numbers were necessary swap with 4 any continue cells, 3 solutions possible.
- n=6 and 24 first natural numbers were necessary swap with 6 any continue cells, 4 solutions possible.
- n=8 and 32 first natural numbers were necessary swap with 8 any continue cells, 5 solutions possible.

Was it possible to see the regularity and infer the following?

- n=10 and 40 first natural numbers was necessary swap with 10 any continue, 6 solutions possible.
- n=12 and 48 first natural numbers was necessary swap with 12 any continue, 7 solutions possible.
- n=14 and 56 first natural numbers was necessary swap with 14 any continue, 8 solutions possible.

And so on. Yes, and was possible to prove these. If it is taken now the sequence {8n},the cases 4, 6, 12, 20, 28 and 36 were automatically excluded.

This problem can be formulated of a more general possible.

"Find general terms $\{x_n\}$ of sequences with two properties. If to n=k and the firsts x_k natural numbers are distributed in 2 rows and $\frac{xk}{2}$ column, then the sums of the columns and rows are equals respectively".

A proof to complete induction to general terms $\{8_n\}$

The first step

First case were already proved

Hypothesis of induction

It possible to see if to P(k)=8k would be 4k cell in each row, the columns have 2,of course. The sum of the 8k first natural number is

$$\frac{8k(8k+1)}{2} = 4k(8k+1) = 32k^2 + 4k$$

The half is $16k^2 + 2k$. However, the row1 sums$16k^2 + 4k$ andthe row2 is $16k^2$and they have2k values of difference, if is exchanged 2k cells, then this value is subtracted and summed to row1 and row2 respectively and the two rows will be equals to $16k^2 + 2k$. Therefore, these properties are valid.

Thesis of induction

To n=k+1, then P(k+1)=8(k+1) = 8k+8.The sum the 8k+8 first natural number is

$$\frac{(8k+8)(8k+8+1)}{2} = (4k+4)(8k+9) = 32k^2 + 68k + 36 = 32(k+1)^2 + 4(k+1)$$

The induction was completed. Besides, as {8n} is divisible for 4, so, the property has been proved as condition sufficient, because the sequence {4n} was excluded, then this condition is not necessary. However, as conjecture is valid. An asking emerged now. Is the divisibility of {Xn} for 8 a property stronger and a conjecture?

Conclusion Part I

The cognitive process to enunciate the problem can be considered in the first moment as a plausible reasoning; because the first approximation to enunciate the problem was achieved by case studies with multiples fails and success, but the proofs were continued until the end. With the proof by complete induction was secure that cognitive process was a plausible reasoning and mathematical demonstration, since the perspective metacognitive, too.

The experience has proved, that, in the construction of a good problem to math competitions is very important that than problem as its solutions could be achieved together, although this generally, in Mathematics Sciences, is not ever possible.

Part II. Implementation of a variant of this problem to students in training

In May 2016, about 40 students from K10 and K11 attended training in the center of Math Olympiad Colombian of the Universidad Antonio Nariño. A sample of 32 students was taken and it was applied in the first test of the final round. The second problem was an adaptation of the problem of the part I.

Find an infinite sequence of numbers $x_1 \ldots x_n$ such that, for every natural n, you can distribute the numbers 1,2,..., $2x_n$ on a board of 2 x X_n, such that the sum of all columns and rows are equal respectively.

The 32 tests were qualified and grouped is two categories:

Excellent	Good and Regular
1	31

The student chosen to case study won numerous honors and awards in Colombian National Olympics, Central Americans and others organized by foreign cities, as Mar del Plata in Argentina. Currently he is studying Mechatronic Engineering. He was qualified as excellent at second problem (he is the third author of this paper), so, was reviewed very carefully, this was necessary to infer thinking type in the process of resolution of the problem.

On the other hand asked what strategy had he been used in the resolution of this problem? He said, literally, *"The strategy I used arose from resolution and analysis of others* *similar problems of number theory, since they help me to think from the key points to solve this problems class, besides the experience gained practicing with a great number of demonstration problems resolved previously".*[1]

He changed first the problem to two columns and x_n rows. This does not change the problem and draw the following board.

(a	)

Description of the student's algorithm

Student's answer is literally translated to English: The sum of 1, 2,…,$2x_n$ is $x_n(2x_n + 1)$, therefore the sum of the rows is $2x_n + 1$. Each column must sum $\frac{x_n(2x_n+1)}{2}$. We know for what $x_n$could be a natural number, must be x_n=2k, then the sum of the rows and columns are 4k+1 and k(4k+1) respectively.

As the sum in each row is 4k+1, then, if it is choose a number n, its partner must necessarily be 4k-n+1 and the couples are defined and it is because doesn't matter in what row put it, only matter the sense, (a;b) not is equal to (b;a).

Now let's look at the difference of these two numbers (n; 4k-n+1) and we can say, without loss of generality that if n≤4k-n+1, then the difference is 4k-2n+1, so if we put that couple first, then the column where it is 4k-n+1 will be larger than the sum of the other 4k-2n+1. If we now place a second pair (m,4k-m+1) only it is interested the difference between them and add it or subtract this value to 4k-2n+1 and will be the new difference between the two columns. If the two columns are equals, is because the process has ended and there are a zero difference.

We know that if the sum of two numbers is odd, then their difference will also be odd. It should begin from 1 pair (2k, 2k+1) up to the required 4k-1 per couple (4k;1). Those odd numbers must be arranged into two groups. If we put the greatest number of the couples in the first column and sum all the differences and the greater number of the other couples in the second column and sum algebraically all differences, then the sums of each column are equal and the sum of the differences is zero.

As 1 + 3 + 5 +... + 4k-1 = $(2k)^2 = 4k^2$, the sum of differences of each column is $2k^2$. If we select in the first row the biggest difference and continue with the minor difference, then the sum of two differences is 4k. If in the second group we choose the second greatest difference and the second smallest difference, then the total of differences is 2(4k) and so on.

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[1] (May 12th 2018). Literal Answers by student about the question: what was the motivation to use this strategy?

As the amount values odd are 2k, then k must be the odd to sum, and k also the amount of odd you must subtract. So, the 2k couples are grouped into two groups, the first group of k couples are placed in the first column the greatest number of the pair and another group of k couples, the bigger number of the partner in the second column. So, if we do the same above process until the end, then they must exist $\frac{k}{2}$, then the quantity of the differences in the first column is $(\frac{k}{2})$ (4k)=2k^2 for which as there are $\frac{k}{2}$ couples, k have to be 2a; so, the sequence must be equal to $\{x_n\}=\{8n\}$.

Application with an example

To n = 4 we have the first 32 natural numbers, and then it has 16 couples. To first group of 8 pairs is:

First	Second	Difference
32	1	+31
17	16	+1
31	2	+29
18	15	+3
30	3	+27
19	14	+5
29	4	+25
20	13	+7
196	68	+128

In the last row is the sum of this group. To second group of the other 8 pairs:

First	Second column	Differenc
5	28	-23
12	21	-9
6	27	-21
11	22	-11
7	26	-19
10	23	-13
8	25	-17
9	24	-15
68	196	-128

In the last row is the sum of this group and when we sum the two groups, then the sums are equal, and difference is zero:

First	Second column	Differenc
264	264	0

Until here, student's response with a good proof. He could generalize a procedure with a direct demonstrative method although this was complicated; besides, he achieved the solution of the problem with the use of analogies in his thinking process.

Applications of the student's algorithm in proofs that in the first part failed

With 36 numbers and the application of the student's algorithm, it is possible to form 18 couples. The first group of 9 pairs must be:

First	Second	Difference
36	1	+35
19	18	+1
35	2	+33
20	17	+3
34	3	+31
21	16	+5
33	4	+29
22	15	+7
32	5	+27
252	81	171

In the last row is the sum of this group. To second group of the other 9 pairs is:

First	Second	Difference
6	31	-25
14	23	-9
7	30	-23
13	24	-11
8	29	-21
12	25	-13
9	28	-19
11	26	-15
10	27	-17
90	243	-143

In the last row is the sum of this group and it can be seen that the sums of the two groups are not equals, and difference is not zero. Therefore, this case was not accomplished with this algorithm neither.

First	Second column	Differenc
342	324	18

Verification of the case to 48 natural numbers with the student's algorithm

The sum of the firsts 48 natural numbers is 1176 it has 24 partners. To first group of 12 pairs or partner is:

First	Second	Difference
48	1	47
25	24	1
47	2	45
26	23	3
46	3	43

27	22	5
45	4	41
28	21	7
44	5	39
29	20	9
43	6	37
30	19	11
438	150	288

In the last row is the sum of this group. To second group of other 12 pairs:

First	Second column	Differenc
18	31	-13
7	42	-35
17	32	-15
8	41	-33
16	33	-17
9	40	-31
15	34	-19
10	39	-29
14	35	-21
11	38	-27
13	36	-23
12	37	-25
150	438	-288

In the last row is the sum of this group. When we sum both groups, then the columns are equal, and the difference is zero:

First	Second column	Differenc
588	588	0

This case had solution with both procedures, besides is divisible for 8 so, this property as necessary condition, confirms that is a good conjecture.

Conclusion Part II

Is this conjecture a theorem? Yes, this is.

Proposition

A condition necessary and sufficient for the general term $\{X_n\}$ of natural numbers to be distributed in two rows and $\frac{n}{2}$ columns with sums of columns and rows equals respectively is that the general term $\{X_n\}$ could be divisible for 8.

Conclusion

The condition is sufficient was proved in part I by complete induction and the necessary condition was of the student as solution.

The cognitive process of the student and of the first author was a mixed between plausible reasoning(Lakatos, I., 1976, Polya, G., 1954) combined with the analogy (Cruz, M., Garcia, M., Rojas, O. and Sigarreta, J., 2016).The first author was inductive with reasoning plausible principally, with a great motivation to generalize the problem. He completed the proof with an insight by comprehension (Canon, C., Garcia, M., 2018).

The cognitive process of the student was deductive in a short time, with a fast solution so, an immediate insight (Canon, C., Garcia, M., 2018), because he had previous knowledge of number theory and could achieve his solution with an analogical process principally.

Biblography

Bellot-Rosado, F., *From a Mathematical Situation to a Problem*, in A. Soifer (Ed.):Competitions for Young Mathematicians,ICME-13,Springer International Publishing (pp 386).Colorado Spring, Co.2017.

Canon, C., Garcia, M., Types of insight present in the solution of math problems in class. *Revista Ciencias Holguin*, 24(2), 10-24, 2018

Cruz, M., Garcia, M., Rojas, O. and Sigarreta, J., Analogies in mathematical problem posing. *Journal of Science Education*, .17(2),84-89, 2016.

Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B., An empirical taxonomy of problem posing processes, Zentralblatt für Didaktik der Mathematik, 37(3), 149-158, 2005.

Engel, A., The creation of mathematical Olympiad Problems. In Newsletter *of the WFNMC,* (5), 1987.

English, L. D., Fox, J. L., & Watters, J. J., Problem posing and solving with mathematical modeling. Teaching Children Mathematics, 12(3), 156-163, 2005.

Falk de Losada, M., *Are Mathematics Competitions changing*, *the Mathematics that is being done and the Way Mathematics Is Done.* In A. Soifer (Ed.): Competitions for Young Mathematicians, ICME-13, Springer International Publishing, (pp 386). Colorado Spring, 2017.

Hersh, R., Introducing Imre Lakatos. *Mathematical Intelligencer*, 1 (3), 148–151, 1978.

Lakatos, I., *Proof and refutations*. Cambridge University Press. New York, 1976.

Lakatos, I., Cauchy and the continuum: The significance of non-standard analysis for the history and philosophy of mathematics. *The Mathematical Intelligencer,* 3, 151–161, mre. 1978.

Kar, T., Özdemir, E., Sabri İpek, A., & Albayrak, M., The relation between the problem posing and problem solving skills of prospective elementary mathematics teachers, World Conference on Educational Sciences, Istanbul: Bahçeşehir University, 2010.

Polya, G., *Mathematics and plausible reasoning*. v.I (Induction and analogy in mathematics). Princeton: Princeton University Press. New Jersey, 1954

Priest, D. J., *A Problem Posing Intervention in the Development of Problem-Solving Competence of Underachieving,* Middle-Year Students. Unpublished doctoral dissertation, Queensland University of Technology, Retrieved April 16, 2010, from: http://eprints.qut.edu.au, 2009.

Silver, E. A., *Foundations of cognitive theory and research for mathematics problem solving instruction.* In A. H. Schoenfeld (Ed.): Cognitive science and mathematics education (pp.3360). Hillsdale, NY: Lawrence Erlbaum. 1987.